ON THE RELATION BETWEEN THE A-POLYNOMIAL AND THE JONES POLYNOMIAL

RĂZVAN GELCA

1. Introduction

In 1984, V. Jones introduced a polynomial invariant of knots [J] through skein relations. Another version of this invariant, the Kauffman bracket [K], was introduced shortly after. Colored versions of these invariants were defined, via quantum groups [RT], and via Jones-Wenzl idempotents [L], [BHMV].

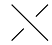
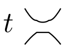
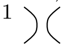
In 1993, Cooper, Culler, Gillet, Long, and Shalen defined a two variable polynomial invariant of knots, the A-polynomial, using the character variety of $SL(2,\mathbb{C})$-representations of the fundamental group of the knot complement. This invariant was generalized in [FGL] to a finitely generated ideal of polynomials in the quantum plane. The construction is done in the context of skein modules, and is based on the fact that the Kauffman bracket skein modules represent deformations of function rings on character varieties [B], [PS], and on the relationship between the skein algebra of the cylinder over a torus and the noncommutative torus [FG].

As shown in [FGL], each element in the noncommutative A-ideal defines a matrix that annihilates the vector whose entries are the colored Jones polynomials of the knot (or, more precisely, the colored Kauffman brackets of the knot; they differ from the colored Jones polynomials by the change of variable $t \mapsto it$). The orthogonality between the rows of the matrix and the vector whose entries are the colored Jones polynomials of the knot has been called the "orthogonality relation (between the Jones polynomial and the A-polynomial)".

In the present paper it is shown that the noncommutative A-ideal together with a finite number (depending on the A-ideal of the knot) of colored Kauffman brackets of the knot determine all other colored Kauffman brackets of the knot. Also, it is shown that, under certain technical conditions on the A-ideal, the noncommutative A-ideal determines all colored Kauffman brackets of the knot. As an example, any knot having the same A-ideal as the unknot, respectively trefoil knot, has the same colored Kauffman brackets as the unknot, respectively trefoil knot.

2. The action of $K_t(\mathbb{T}^2 \times I)$ on $K_t(\mathbb{D}^2 \times I)$

The Kauffman bracket skein module of the three manifold M is defined in the following way. Let $\mathbb{C}[t,t^{-1}]\mathcal{L}$ be the $\mathbb{C}[t,t^{-1}]$-module freely spanned

by the isotopy classes of framed links in M including the empty link, and let $\mathcal{S}$ be the submodule spanned by the relations  $-\,t$  $-\,t^{-1}$  and  $+\,t^2+t^{-2}$. The Kauffman bracket skein module of M is $K_t(M)=\mathbb{C}[t]/\mathcal{S}$.

In the case where M is the cylinder over a surface, $K_t(M)$ has a natural algebra structure, with product defined by placing one link on top of another. If M is a manifold with boundary, the operation of gluing a cylinder to the boundary induces a $K_t(\partial M\times I)$-module structure on $K_t(M)$. As an example it is known that the Kauffman bracket skein algebra of the cylinder over an annulus (i.e., that of the solid torus), is $\mathbb{C}[t,t^{-1},\alpha]$, where α is the curve that runs once around the annulus and has framing parallel to the annulus.

Another, more complicated example is that of the Kauffman bracket skein algebra of $K_t(\mathbb{T}^2\times I)$. Its multiplication rule and action on the skein module of the solid torus are described by means of two families of Chebyshev polynomials, $\{T_n\}_{n\in\mathbb{Z}}$ defined by $T_0=2, T_1=x$, $T_{n+1}=xT_n-T_{n-1}$ for $n\in\mathbb{Z}$ and $\{S_n\}_{n\in\mathbb{Z}}$ defined by $S_0=2, S_1=x$, $S_{n+1}=xS_n-S_{n-1}$ for $n\in\mathbb{Z}$. Let p and q be two integers with $p=np'$, $q=nq'$, p', q' coprime. We define $(p,q)_T=T_n((p',q'))$, where (p',q') is the corresponding curve on the torus, with framing parallel to the torus, and its powers are defined by parallel copies. The elements $(p,q)_T$, $p\geq 0, q\in\mathbb{Z}$ span $K_t(M)$ as a $\mathbb{C}[t,t^{-1}]$-module. In [FG] we proved the following product-to-sum formula

$$(p,q)_T*(r,s)_T=t^{\left|\begin{smallmatrix}p&q\\r&s\end{smallmatrix}\right|}(p+r,q+s)_T+t^{-\left|\begin{smallmatrix}p&q\\r&s\end{smallmatrix}\right|}(p-r,q-s)_T.$$

As a consequence of this formula, the Kauffman bracket skein algebra of the cylinder over a torus is isomorphic to the subalgebra of the noncommutative torus generated by noncommutative cosines. Let us recall that the algebra of trigonometric polynomials in the noncommutative torus is $\mathbb{C}_t[l,l^{-1},m,m^{-1}]$, with multiplication $*$, satisfying $l*m=t^2m*l$. The elements $e_{p,q}=t^{-pq}l^pm^q$ are the noncommutative exponentials; they satisfy

$$e_{p,q}*e_{r,s}=t^{\left|\begin{smallmatrix}p&q\\r&s\end{smallmatrix}\right|}e_{p+r,q+s}.$$

The noncommutative cosines are $\frac{1}{2}(e_{p,q}+e_{-p,-q})$. The map $(p,q)_T\to e_{p,q}+e_{-p,-q}$ gives the isomorphism between $K_t(\mathbb{T}^2\times I)$ and the algebra of noncommutative cosines.

Let K be a knot in S^3, and M the complement of a regular neighborhood of K. Recall the left action of $K_t(\mathbb{T}^2\times I)$ on $K_t(M)$. The peripheral ideal of K is the left ideal of $K_t(\mathbb{T}^2\times I)$ which annihilates the empty link. The noncommutative A-ideal of K, denoted by $\mathcal{A}_t(K)$ is the left ideal obtained by extending $I_t(K)$ to $\mathbb{C}_t[l,l^{-1},m,m^{-1}]$ then contracting it to $\mathbb{C}_t[l,m]$. As explained in [FGL], this is a noncommutative generalization of the A-polynomial. The A-polynomial is obtained by setting $t=-1$, replacing l and m by $-l$ and $-m$ and taking the generator of the radical of the one-dimensional part of the A-ideal (divided by $(l-1)$).

There is a left and a right action of $K_t(\mathbb{T}^2\times I)$ on $K_t(\mathbb{D}^2\times S^1)$, one for the positive, the other one for the negative orientation of the boundary torus. To understand them, let us denote by $x_{p,q}$ the image in $K_t(\mathbb{D}^2\times S^1)$

of $(p,q)_T$ on the boundary torus (with the positive orientation). It is not hard to see that $x_{0,q} = (-t^2)^q + (-t^{-2})^q$ and the product-to-sum formula yields

$$x_{p+1,q} = t^{-q}(1,0) \cdot x_{p,q} - t^{-2q} x_{p-1,q}.$$

The second order recurrence relation for $t^{pq}x_{p,q}$ has fixed coefficients, and hence a formula for the general term can be found. It is

$$x_{p,q} = t^{-pq}((-t^{-2})^q S_p(\alpha) - (-t^2)^q S_{p-2}(\alpha)).$$

Lifting the skeins $T_n(\alpha)$ to the boundary torus and using the product-to-sum formula we get the following

Lemma 1. *The left action is described by*

$$\begin{aligned} (p,q)_T \cdot T_n(\alpha) &= t^{-(2n+p)q}[(-t^{-2})^q S_{n+p}(\alpha) - (-t^{-2})^q S_{n+p-2}(\alpha)] \\ &\quad + t^{(2n-p)q}[(-t^{-2})^q S_{p-n}(\alpha) - (-t^2)^q S_{p-n-2}(\alpha)] \end{aligned}$$

while the right action is given by

$$\begin{aligned} T_n(\alpha) \cdot (p,q)_T &= (p,-q)_T \cdot T_n(\alpha) \\ &= t^{(2n+p)q}[(-t^2)^q S_{p+n}(\alpha) - (-t^{-2})^q S_{p+n-2}(\alpha)] \\ &\quad + t^{-(2n-p)q}[(-t^2)^q S_{p-n}(\alpha) - (-t^{-2})^q S_{p-n-2}(\alpha)]. \end{aligned}$$

3. The results

Gluing a solid torus to the complement M of a regular neighborhood of a knot K, in such a way that the longitude is glued to the longitude and the meridian to the meridian, induces a pairing

$$K_t(\mathbb{D} \times S^1) \times K_t(M) \to \mathbb{C}[t, t^{-1}].$$

The basis $\{S_n(\alpha)\}_n$ induces a family of functionals $< S_n(\alpha), \cdot >$, $n = 0, 1, 2, \ldots$. If we denote by $\emptyset$ the empty link, then

$$< S_n(\alpha), \emptyset >= \kappa_n(K),$$

where $\kappa_n(K)$ is the nth colored Kauffman bracket of K with zero framing [L], [T] (the nth colored Kauffman bracket is a "twisted" version of the nth colored Jones polynomial as defined in [RT]). Indeed, the recurrence relation for S_n shows that the link in S^3 obtained from the pairing is K colored by the Jones-Wenzl idempotent.

The pairing is compatible with the actions of $K_t(\mathbb{T}^2 \times I)$ on both modules, i.e. $< u \cdot (p,q)_T, v >=< u, (p,q)_T \cdot v >$ for any skeins u and v. In particular, if a is in the peripheral ideal $I_t(K)$ of K, then $< u \cdot a, \emptyset >= 0$. So, if

$u = T_n(\alpha)$, and $a = \sum_i c_i(p_i, q_i)_T$, then by Lemma 1,

$$
\begin{aligned}
&< a \cdot T_n(\alpha), \emptyset >= \sum_i c_i (t^{p_i q_i} (t^{(2n+p_i)q_i} [(-t^2)^{q_i} < S_{p_i+n}(\alpha), \emptyset > \\
&-(-t^{-2})^{q_i} < S_{p_i+n-2}(\alpha), \emptyset >] \\
&+t^{-(2n-p_i)q_i}[(-t^2)^{q_i} < S_{p_i-n}(\alpha), \emptyset > -(-t^{-2})^{q_i} < S_{p_i-n-2}(\alpha), \emptyset >]) \\
= \ &\sum_i c_i (t^{(2n+p_i)q_i} [(-t^2)^{q_i} \kappa_{p_i+n}(K) - (-t^{-2})^{q_i} \kappa_{p_i+n-2}(K)] \\
&+t^{-(2n-p_i)q_i}[(-t^2)^{q_i} \kappa_{p_i-n}(K) - (-t^{-2})^{q_i} \kappa_{p_i-n-2}(K)]).
\end{aligned}
$$

This relation has been called the orthogonality relation in [FGL] since it expresses the orthogonality between the vector with entries equal to the colored Kauffman brackets of the knot and the rows of the matrix of the linear transformation induced by a between the module $K_t(\mathbb{D} \times I)$ with basis $\{T_n(\alpha)\}_n$ and the same module with basis $\{S_n(\alpha)\}_n$. Since a arises from an element in the noncommutative A-ideal (through an extension and a contraction), orthogonality expresses a relationship between the the elements of the A-ideal and the vector whose entries are the colored Kauffman brackets.

Theorem 1. *For every knot K there is a number $\nu(K)$ such that if K' is a knot with $\mathcal{A}_t(K) = \mathcal{A}_t(K')$ and $\kappa_j(K) = \kappa_j(K')$ for $j = 1, 2, \cdots, \nu(K)$, then $\kappa_j(K) = \kappa_j(K')$ for all j. Moreover, $\nu(K)$ depends only on the A-ideal of K.*

Proof. Choose $a = \sum_j c_j(p_j, q_j)_T$ some element in $I_t(K)$, let p be the maximum of p_j and assume $p_j = p$ if $j = 1, 2, \cdots, m$, $p_j \neq p$ if $j > m$. Then, coefficient of κ_{n+p} in the orthogonality relation written for a is

$$\sum_{j=1}^{m} c_j (-1)^{q_j} t^{(2n+2+p)q_j}$$

Since the q_j appearing in this expression are distinct (the p_i's being the same), this expression is identically equal to zero only for finitely many n. Hence the orthogonality relation provides a recurrence relation that determines uniquely κ_n for large n. □

As the result below shows, in certain situations the A-ideal determines the colored Kauffman brackets of the knot.

Theorem 2. *Assume that K is a knot with the property that $\mathcal{A}_t(K)$ contains a polynomial $\sum_{p,q} \gamma_{p,q} l^p m^q$ of degree 2 in l such that there exists no $n \geq 0$ for which the expression $\sum_q \gamma_{2,q}(-1)^q t^{(2n+2)q}$ is identically equal to zero. Then for any knot K' with the property that $\mathcal{A}_t(K) = \mathcal{A}_t(K')$, it follows that $\kappa_n(K) = \kappa_n(K')$ for all $n = 1, 2, 3, \ldots$.*

Proof. The polynomial gives rise to an element $a = \sum_i c_i(1, q_i)_T + u$ in $I_t(K)$, with $c_i = t^{q_i} \gamma_{2,q_i}$ and u a polynomial in $(0, 1)$. By Lemma 1, $T_n(\alpha) \cdot u$

is of the form $\lambda S_n(\alpha)+\mu S_{n-2}(\alpha)$, $\lambda,\mu\in\mathbb{C}[t,t^{-1}]$. On the other hand, the same lemma shows that

$$T_n(\alpha)\cdot\sum_i c_i(1,q_i)_T=\sum_i c_i[(-t)^{(2n+3)q_i}S_{n+1}(\alpha)$$
$$-(-t)^{(2n-1)q_i}S_{n-1}(\alpha)+(-t)^{(-2n+3)}S_{1-n}(\alpha)-(-t)^{(-2n-1)q_i}S_{-n-1}(\alpha)].$$

Since $S_{-k}=-S_{k-2}$ for all k, this is further equal to

$$\sum_i c_i[(-t)^{(2n+3)q_i}S_{n+1}(\alpha)-[(-t)^{(2n-1)q_i}+(-t)^{(-2n-1)q_i}]S_{n-1}(\alpha)$$
$$+(-t)^{(-2n+3)q_i}S_{n-3}(\alpha)].$$

Hence the orthogonality relation applied to a yields a $\mathbb{C}[t,t^{-1}]$-linear equation in $\kappa_{n+1}(K),\kappa_n(K),\kappa_{n-1}(K),\kappa_{n-2}(K)$, and $\kappa_{n-3}(K)$. For $n\geq 1$, the coefficient of $\kappa_{n+1}(K)$ is $\sum_i c_i(-t)^{(2n+3)q_i}$, and the condition from the statement translates to the fact that for no n this is identically equal to zero. Therefore, the orthogonality relation provides a recursion that uniquely determines $\kappa_n(K)$ from $\kappa_1(K)$.

For $n=0$, since $\kappa_0(K)=1$, $\kappa_{-1}(K)=0$, $\kappa_{-2}(K)=-1$, $\kappa_{-3}(K)=-\kappa_1(K)$, the orthogonality relation gives a linear equation in $\kappa_1(K)$. The coefficient of $\kappa_1(K)$ is $2\sum_i c_i(-t)^{3q_i}$. Again this is not equal to zero. So the equation can be solved uniquely for $\kappa_1(K)$. It follows that the A-ideal determines the colored Kauffman brackets of the knot, and we are done. □

Observe that the degree in l of any polynomial in $\mathcal{A}_t(K)$ is at least 2.

4. Examples

4.1. **The unknot.** The A-ideal of the unknot is generated by $(l+t^2)(l+t^{-2})$ and $lm^2(l+t^2)+t^2(l+t^{-2})$ [FGL]; hence it satisfies the conditions in Theorem 2. The orthogonality relation for $(l+t^2)(l+t^{-2})$, that is, for $(1,0)_T+t^2+t^{-2}\in I_t(K)$, gives

$$\kappa_0(K)=1,\quad \kappa_1(K)=-t^2-t^{-2},$$
$$\kappa_{n+1}(K)=(-t^2-t^{-2})\kappa_n(K)-\kappa_{n-1}(K),\quad n\geq 1.$$

From this we obtain the well known formula

$$\kappa_n(K)=(-1)^n(t^{2n+2}-t^{-2n-2})/(t^2-t^{-2}).$$

The orthogonality relation for the other element leads to a different recurrence relation with the same solution.

4.2. **The trefoil.** The A-ideal of the left-handed trefoil is generated by $[m^4(l+t^{10})-t^{-4}(l+t^2)](l-t^6m^6)$, $(l+t^{24})(l+t^{10})(l+t^2)(l-t^6m^6)$ and $(m^2-t^{-22})(l+t^{10})(l+t^2)(l-t^6m^6)$ [G]. A quick look at the element $[m^4(l+t^{10})-t^{-4}(l+t^2)](lm^6-t^6)$ shows that the conditions in the statement of Theorem 2 are fulfilled. This element corresponds to

$$(1,-5)_T-t^{-8}(1,-1)_T+t^3(0,5)_T-t(0,1)_T$$

in the peripheral ideal. The orthogonality relation produces the following recursion

$$\begin{gathered}(-t^{-10n-15}+t^{-2n-11})\kappa_{n+1}(K)+(-t^{10n+7}-t^{-10n-13}+t^{2n+3}\\+t^{-2n-1})\kappa_n(K)+(t^{-10n+5}-t^{10n+5}-t^{-2n-7}+t^{2n-7})\kappa_{n-1}(K)+(t^{10n-13}\\+t^{-10n+7}-t^{2n-1}-t^{-2n+3})\kappa_{n-2}(K)+(t^{10n-15}-t^{2n-11})\kappa_{n-3}(K)=0.\end{gathered}$$

In particular, for $n=0$,

$$(t^{-11}-t^{-15})\kappa_1(K)-t^7-t^{-13}+t^3+t^{-1}=0,$$

and hence $\kappa_1(K)=t^{18}-t^{10}-t^6-t^2$, the well known formula for the Kauffman bracket of the trefoil knot with framing zero.

References

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology **31**(1992), 685–699.

[B] D. Bullock, *Rings of $SL_2\mathbb{C}$-characters and the Kauffman bracket skein module*, Comment. Math. Helv. **72**(1997), no. 4, 521–542.

[CCGLS] D. Cooper, M. Culler, H. Gillett, D.D. Long, P.B. Shalen, *Plane Curves associated to character varieties of 3-manifolds*, Inventiones Math. **118**, pp. 47-84 (1994).

[FG] C. Frohman, R. Gelca, *Skein Modules and the Noncommutative Torus*, Transactions of the AMS, to appear (xxx.lanl.gov/math.QA-9806107).

[FGL] C. Frohman, R. Gelca, W. Lofaro, *The A-polynomial from the noncommutative viewpoint*, preprint.

[G] R. Gelca, *Noncommutative trigonometry and the A-polynomial of the trefoil knot*, preprint.

[J] V.F.R. Jones, *Polynomial invariants of knots via von Neumann algebras*, Bull. Amer. Math. Soc., **12**(1985), 103–111.

[K] L. Kauffman, *States models and the Jones polynomial*, Topology, **26**(1987), 395–407.

[L] W.B.R. Lickorish, *The skein method for three-manifold invariants*, J. Knot Theor. Ramif., **2**(1993) no. 2, 171–194.

[PS] J. H. Przytycki and A. Sikora, *Skein algebra of a group*, Banach Center Publ. 42.

[RT] N.Yu. Reshetikhin, V.G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Inventiones Math., **103**(1991), 547–597.

[T] V.G. Turaev, *Quantum invariants of Knots and 3-manifolds*, de Gruyter Studies in Mathematics, de Gruyter, Berlin–New York, 1994.

Department of Mathematics,University of Michigan, Ann Arbor, MI 48109, USA and Institute of Mathematics of the Romanian Academy, Bucharest, Romania

E-mail address: rgelca@math.lsa.umich.edu